

NON-DISPERSIVE VANISHING AND BLOW UP AT INFINITY FOR THE ENERGY CRITICAL NONLINEAR SCHRÖDINGER EQUATION IN \mathbb{R}^3

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Dedicated to the memory of Vladimir Savelievich Buslaev

ABSTRACT. We consider the energy critical focusing nonlinear Schrödinger equation $i\psi_t = -\Delta\psi - |\psi|^4\psi$ in \mathbb{R}^3 , and prove, for any ν and α_0 sufficiently small, the existence of radial finite energy solutions of the form $\psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + e^{i\Delta t}\zeta^* + o_{\dot{H}^1}(1)$ as $t \rightarrow +\infty$, where $\alpha(t) = \alpha_0 \ln t$, $\lambda(t) = t^\nu$, $W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}$ is the ground state, and ζ^* is arbitrary small in \dot{H}^1 .

1. INTRODUCTION

1.1. Setting of the problem and statement of the result. In this paper we consider the energy critical focusing nonlinear Schrödinger equation

$$(1.1) \quad \begin{aligned} i\psi_t &= -\Delta\psi - |\psi|^4\psi, \quad x \in \mathbb{R}^3, \\ \psi|_{t=0} &= \psi_0 \in \dot{H}^1(\mathbb{R}^3). \end{aligned}$$

Cauchy problem (1.1) is locally well posed and the solutions during their life span satisfy conservation of energy:

$$(1.2) \quad E(\psi(t)) \equiv \int (|\nabla\psi(x, t)|^2 - \frac{1}{3}|\psi(x, t)|^6) dx = E(\psi_0).$$

The problem is energy critical in the sense that both (1.1) and (1.2) are invariant with respect to the scaling $\psi(x, t) \rightarrow \lambda^{1/2}\psi(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}_+$. For \dot{H}^1 small data one has global existence and scattering. In the case of large data blow up may occur. Indeed, the classical virial identity

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(x, t)|^2 dx = 8 \int (|\nabla\psi(x, t)|^2 - |\psi(x, t)|^6) dx$$

shows that if $x\psi_0 \in L^2(\mathbb{R}^3)$ and $E(\psi_0) < 0$, then the solution breaks down in finite time.

Furthermore, Eq. (1.1) admits an explicit stationary solution (ground state):

$$W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}, \quad \Delta W + W^5 = 0,$$

so that scattering cannot always occur even for solutions that exist globally in time.

The ground state W is known to play an important role in the dynamics of (1.1). It was proved by Kenig and Merle [6] that $E(W)$ is an energy threshold for the dynamics in the following sense. If ψ_0 is radial and $E(\psi_0) < E(W)$ then

- (i) the solution of (1.1) is global and scatters to zero as a free wave in both directions, provided $\|\nabla\psi_0\|_{L^2} < \|\nabla W\|_{L^2}$;
- (ii) the solution blows up in finite time in both direction, provided $\psi_0 \in L^2$ and $\|\nabla\psi_0\|_{L^2} > \|\nabla W\|_{L^2}$.

The behavior of radial solutions with critical energy $E(\psi_0) = E(W)$ was classified by Duyckaerts and Merle [5]. In this case, in addition to the finite time blow up and scattering to zero (and W itself), one has the existence of solutions that converge as $t \rightarrow \infty$ to a rescaled ground state. In the case of energy slightly greater than $E(W)$ the dynamics is expected to be more rich and to include the solutions that as $t \rightarrow \infty$ behave like $e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x)$ with fairly general $\alpha(t)$ and $\lambda(t)$. For a closely related model of the critical wave equation, the existence of this type of solutions with $\lambda(t) \rightarrow \infty$ (blow up at infinity) and $\lambda(t) \rightarrow 0$, $t\lambda(t) \rightarrow \infty$ (non-dispersive vanishing) was recently proved by Donninger and Krieger [4]. Our objective in this paper is to obtain an analogous result for NLS (1.1). More precisely, we prove the following.

Theorem 1.1. *There exists $\beta_0 > 0$ such that for any $\nu, \alpha_0 \in \mathbb{R}$ with $|\nu| + |\alpha_0| \leq \beta_0$ and any $\delta > 0$ there exist $T > 0$ and a radial solution $\psi \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ to (1.1) of the form:*

$$(1.3) \quad \psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x, t),$$

where $\lambda(t) = t^\nu$, $\alpha(t) = \alpha_0 \ln t$, and $\zeta(t)$ verifies:

$$(1.4) \quad \begin{aligned} \|\zeta(t)\|_{\dot{H}^1 \cap \dot{H}^2} &\leq \delta, \\ \|\zeta(t)\|_{L^\infty} &\leq Ct^{-\frac{1+\nu}{2}}, \\ \|\langle \lambda(t)x \rangle^{-1} \zeta(t)\|_{L^\infty} &\leq Ct^{-1-\frac{3}{2}\nu}, \end{aligned}$$

for all $t \geq T$. The constants C here and below are independent of ν, α_0 and δ .

Furthermore, there exists $\zeta^* \in \dot{H}^s$, $\forall s > \frac{1}{2} - \nu$, such that, as $t \rightarrow +\infty$, $\zeta(t) - e^{it\Delta}\zeta^* \rightarrow 0$ in $\dot{H}^1 \cap \dot{H}^2$.

Remark 1.2. Theorem 1.1 remains valid, in fact, with \dot{H}^2 replaced by \dot{H}^k for any $k \geq 2$ (with β_0 depending on k).

Remark 1.3. The restriction on ν and α_0 that appears in Theorem 1.1 seems to be technical. One might expect the same result to be true for any $\nu > -1/2$ and any $\alpha_0 \in \mathbb{R}$.

Remark 1.4. The solutions we construct to prove the theorem belong, in fact, to $\dot{H}^{\frac{1}{2}-\nu+}$.

Remark 1.5. Using the techniques developed in this paper one can prove the existence of radial finite time blow up solutions of the form $\psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x, t)$, $\lambda(t) = (T - t)^{-1/2-\nu}$, $\alpha(t) = \alpha_0 \ln(T - t)$, where $\zeta(t)$ is arbitrary small in $\dot{H}^1 \cap \dot{H}^2$ and $\nu > 1$, $\alpha_0 \in \mathbb{R}$ can be chosen arbitrarily. For the critical wave equation an analogous result was proved by Krieger, Schlag, Tataru in [8], see also [9] for a similar construction in the context of the critical Schrödinger map equation.

1.2. Outline of the paper. The paper is organized as follows. In Section 2 we construct (Prop. 2.1) a sufficiently good approximate solution of (1.1) very much in the spirit of [4], [8], [9]. In Section 3 we build up an exact solution by solving the problem for the small remainder with zero initial data at infinity, the main technical tool of the construction being some suitable energy type estimates for the linearized evolution. These estimates are proved in Section 4.

2. APPROXIMATE SOLUTIONS

In this section we prove the following result.

Proposition 2.1. *For any ν and α_0 sufficiently small and any $0 < \delta \leq 1$ there exists a radial approximate solution $\psi^{ap} \in C^\infty(\mathbb{R}^3, \mathbb{R}_+^*)$ of (1.1) such that the following holds for $t \geq T$ with some $T = T(\nu, \alpha_0, \delta) > 0$.*

(i) ψ^{ap} has the form: $\psi^{ap}(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)(W(\lambda(t)x) + \chi^{ap}(\lambda(t)x, t))$, where $\chi^{ap}(y, t)$, $y = \lambda(t)x$, verifies

$$(2.1) \quad \|\chi^{ap}(t)\|_{\dot{H}^k} \leq C\delta^{\nu+k-1/2}t^{-\nu(k-1)}, \quad k = 1, 2,$$

$$(2.2) \quad \|\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-(1+2\nu)/2},$$

$$(2.3) \quad \| |y|^{-1}\chi^{ap}(t) \|_{L^\infty} + \|\nabla \chi^{ap}(t)\|_{L^\infty} \leq Ct^{-1-2\nu},$$

$$(2.4) \quad \| |y|^{-2}\chi^{ap}(t) \|_{L^\infty} + \| |y|^{-1}\nabla_y \chi^{ap}(t) \|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu},$$

$$(2.5) \quad \|\nabla^2 \chi^{ap}(t)\|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu}.$$

Furthermore, there exists $\zeta^* \in \dot{H}^s$, for any $s > \frac{1}{2} - \nu$, such that, as $t \rightarrow +\infty$, $e^{i\alpha(t)}\lambda^{1/2}(t)\chi^{ap}(\lambda(t)\cdot, t) - e^{it\Delta}\zeta^* \rightarrow 0$ in $\dot{H}^1 \cap \dot{H}^2$.

(ii) The corresponding error $R = -i\psi_t^{ap} - \Delta\psi^{ap} - |\psi^{ap}|^4\psi^{ap}$ satisfies

$$(2.6) \quad \|R(t)\|_{\dot{H}^k} \leq t^{-(2+\frac{1}{8})(1+2\nu)+\nu(k+1)}, \quad k = 0, 1, 2.$$

The construction of $\psi^{ap}(t)$ will be achieved by considering separately the three regions that correspond to three different space scales: the inner region with the scale $t^\nu|x| \lesssim 1$, the self-similar region where $|x| = O(t^{1/2})$, and, finally, the remote region where $|x| = O(t)$. In the inner region the solution will be constructed as a perturbation of the profile $e^{i\alpha_0 \ln t}t^{\nu/2}W(t^\nu x)$. The self-similar and remote regions are the regions where the solution is small and is described essentially by the linear equation $i\psi_t = -\Delta\psi$. In the self-similar region the profile of the solution will be determined uniquely by

the matching conditions coming out from the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin is prescribed by the matching procedure.

2.1. The inner region. We start by considering the inner region $0 \leq t^\nu |x| \leq 10t^{1/2+\nu-\epsilon_1}$ with $0 < \epsilon_1 < 1/2 + \nu$ to be fixed later. Writing $\psi(x, t)$ as $\psi(x, t) = e^{i\alpha(t)} \lambda^{1/2}(t) u(\rho, t)$, $\rho = \lambda(t)|x|$, we get from (1.1)

$$(2.7) \quad it^{-2\nu} u_t - \alpha_0 t^{-(1+2\nu)} u + i\nu t^{-(1+2\nu)} \left(\frac{1}{2} + \rho \partial_\rho \right) u = -\Delta u - |u|^4 u.$$

Write $u(\rho, t) = W(\rho) + \chi(\rho, t)$. Then $\vec{\chi}(t) = \begin{pmatrix} \chi(t) \\ \bar{\chi}(t) \end{pmatrix}$ solves

$$(2.8) \quad it^{-2\nu} \vec{\chi}_t = H \vec{\chi} + \mathcal{N}(\chi),$$

where

$$H = -\Delta \sigma_3 - 3W^4 \sigma_3 - 2W^4 \sigma_3 \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{N}(\chi) = \begin{pmatrix} N(\chi) \\ -\overline{N(\chi)} \end{pmatrix}, \quad N(\chi) = N_0 + N_1(\chi) + N_2(\chi),$$

$$N_0 = \alpha_0 t^{-(1+2\nu)} W - i\nu t^{-(1+2\nu)} W_1, \quad W_1(\rho) = \left(\frac{1}{2} + \rho \partial_\rho \right) W(\rho)$$

$$N_1(\chi) = \alpha_0 t^{-(1+2\nu)} \chi - i\nu t^{-(1+2\nu)} \left(\frac{1}{2} + \rho \partial_\rho \right) \chi,$$

$$N_2(\chi) = -|W + \chi|^4 (W + \chi) + W^5 + 3W^4 \chi + 2W^4 \bar{\chi}.$$

We look for a solution to (2.8) of the form

$$(2.9) \quad \chi(\rho, t) = \sum_{k=1}^{\infty} t^{-k(1+2\nu)} \chi_k(\rho).$$

Substituting (2.9) into (2.8) and identifying the terms with the same powers of t we get the following system for $\{\chi_k\}_{k \geq 1}$:

$$(2.10) \quad H \vec{\chi}_k = \mathcal{D}_k, \quad k \geq 1,$$

$$\text{where } \mathcal{D}_k = \begin{pmatrix} D_k \\ -\overline{D_k} \end{pmatrix},$$

$$D_1 = -\alpha_0 W + i\nu W_1,$$

$$D_k = D_k^{(1)} + D_k^{(2)}, \quad k \geq 2,$$

$D_k^{(1)}$ and $D_k^{(2)}$ being contributions of $it^{-2\nu}\chi_t - N_1(\chi)$ and $-N_2(\chi)$ respectively:

$$D_k^{(1)} = -i(1+2\nu)(k-1)\chi_{k-1} - \alpha_0\nu\chi_{k-1} + i\nu\left(\frac{1}{2} + \rho\partial_\rho\right)\chi_{k-1},$$

$$N_2(\chi) = -\sum_{k=2}^{\infty} t^{-k(1+2\nu)} D_k^{(2)}(\rho).$$

Note that D_k depends on χ_p , $1 \leq p \leq k-1$ only:

$$D_k = D_k(\rho; \chi_p, 1 \leq p \leq k-1).$$

We subject (2.10) to zero initial conditions at 0: $\chi_k(0) = \partial_\rho\chi_k(0) = 0$.

Lemma 2.2. *System (2.10) has a unique solution $\{\chi_k\}_{k \geq 1}$ verifying:*

- i) for any $k \geq 1$, χ_k is a C^∞ function that has an even Taylor expansion at $\rho = 0$ that starts at order $2k$;*
- ii) as $\rho \rightarrow +\infty$, χ_k , $k \geq 1$, has the following asymptotic expansion*

$$(2.11) \quad \chi_k(\rho) = \sum_{l=0}^k \sum_{j \leq 2k-2l-1} \alpha_{l,j}^{(k)} (\ln \rho)^l \rho^j,$$

with some coefficients $\alpha_{l,j}^{(k)}$ verifying $\alpha_{k,2m}^{(k)} = 0$ for all k, m . The asymptotic expansion (2.11) can be differentiated any number of times with respect to ρ .

Proof. It will be convenient for us to rewrite (2.10) as

$$(2.12) \quad L_+ v_k^+ = G_k^+, \quad L_- v_k^- = G_k^-, \quad k \geq 1,$$

where

$$\begin{aligned} v_k^+ &= \operatorname{Re} \chi_k, & v_k^- &= \operatorname{Im} \chi_k, \\ G_k^+ &= \operatorname{Re} D_k, & G_k^- &= \operatorname{Im} D_k, \\ L_+ &= -\Delta - 5W^4, & L_- &= -\Delta - W^4. \end{aligned}$$

For $k = 1$ (2.12) gives

$$(2.13) \quad L_+ v_1^+ = -\alpha_0 W, \quad L_- v_1^- = \nu W_1.$$

The homogeneous equation $L_\pm f = 0$ has two explicit solutions Φ_\pm, Θ_\pm given by

$$(2.14) \quad \begin{aligned} \Phi_-(\rho) &= W(\rho), & \Theta_-(\rho) &= \left(1 + \frac{\rho^2}{3}\right)^{-1/2} \left(\frac{\rho}{3} - \frac{1}{\rho}\right), \\ \Phi_+(\rho) &= W_1(\rho), & \Theta_+(\rho) &= -2 \left(1 + \frac{\rho^2}{3}\right)^{-3/2} \left(\frac{1}{\rho} - 2\rho + \frac{\rho^3}{9}\right). \end{aligned}$$

Therefore, solving (2.13) with zero initial conditions at the origin we obtain

$$(2.15) \quad \begin{aligned} v_1^+(\rho) &= \alpha_0 \int_0^\rho s^2 (\Theta_+(\rho)\Phi_+(s) - \Theta_+(s)\Phi_+(\rho)) W(s) ds, \\ v_1^-(\rho) &= -\nu \int_0^\rho s^2 (\Theta_-(\rho)\Phi_-(s) - \Theta_-(s)\Phi_-(\rho)) W_1(s) ds. \end{aligned}$$

Since W, W_1 are C^∞ even functions, v_1^+ and v_1^- are also C^∞ functions with even Taylor expansion at $\rho = 0$ that starts at order 2. Furthermore, the asymptotic expansions of v_1^+ and v_1^- as $\rho \rightarrow +\infty$ can be obtained directly from (2.15). As claimed, one has

$$v_1^+(\rho) + iv_1^-(\rho) = \sum_{j \leq 1} \alpha_{0,j}^{(1)} \rho^j + \sum_{j \leq 0} \alpha_{1,j}^{(1)} \rho^{2j-1} \ln \rho, \quad \text{as } \rho \rightarrow +\infty.$$

We next proceed by induction. Let us consider $k > 1$ and assume that we have found $\chi_i, i = 1, \dots, k-1$, that verify i), ii). Then one can easily check that D_k is an even C^∞ function with a Taylor series at 0 starting at order $2(k-1)$ and as $\rho \rightarrow +\infty$, D_k admits an asymptotic expansion of the form

$$D_k(\rho) = \sum_{l=0}^{k-1} \sum_{j \leq 2k-2l-3} d_{j,l}^{(k)} (\ln \rho)^l \rho^j + (\ln \rho)^k \sum_{j \leq -5} d_{j,k}^{(k)} \rho^j,$$

where $d_{-2,k-1}^{(k)} = 0$ and $d_{2m,k}^{(k)} = 0, \forall m$. Therefore, solving $L_\pm v_k^\pm = G_k^\pm$ with zero conditions at $\rho = 0$ we get a C^∞ even solution v_k^\pm which is $O(\rho^{2k})$ at the origin. Finally, the asymptotic expansion at infinity follows directly from the representation

$$v_k^\pm(\rho) = - \int_0^\rho s^2 (\Theta_\pm(\rho) \Phi_\pm(s) - \Theta_\pm(s) \Phi_\pm(\rho)) G_k^\pm(s) ds.$$

□

Remark 2.3. Clearly, for any k , χ_k is a polynomial with respect to α_0 and ν of the form

$$\chi_k = \sum_{1 \leq m+n \leq k} \alpha_0^m \nu^n \chi_{m,n}^k(\rho),$$

where the coefficients $\chi_{m,n}^k$ are C^∞ functions of ρ with an even Taylor expansion at 0 that starts at order $2k$. As $\rho \rightarrow +\infty$, $\chi_{m,n}^k$ admits an asymptotic expansion of the form (2.11).

For any $N \geq 2$, define

$$\chi^{(N)}(\rho, t) = \sum_{k=1}^N t^{-k(1+2\nu)} \chi_k(\rho).$$

It follows from our construction that $\chi^{(N)}$ verifies

$$(2.16) \quad \left| \rho^{-k} \partial_\rho^l (-it^{-2\nu} \bar{\chi}_t^{(N)} + H \bar{\chi}^{(N)} + \mathcal{N}(\chi^{(N)})) \right| \leq C_{N,l,k} t^{-(N+1)(1+2\nu)} < \rho >^{2N-1-l-k},$$

for any $k, l \in \mathbb{N}$, $k+l \leq 2N$, $0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1}$, $t \geq 1$.

Fix $N = 27$, $\epsilon_1 = \frac{1+2\nu}{27}$,¹ and set

$$u_{in}^{ap} = W + \chi_{in}^{ap}, \quad \chi_{in}^{ap} = \chi^{(27)},$$

$$\mathcal{R}_{in} = -it^{-2\nu} \partial_t u_{in}^{ap} - \Delta u_{in}^{ap} + \alpha_0 t^{-1-2\nu} u_{in}^{ap} - i\nu t^{-1-2\nu} \left(\frac{1}{2} + \rho \partial_\rho \right) u_{in}^{ap} - |u_{in}^{ap}|^4 u_{in}^{ap}.$$

As a direct consequence of Lemma 2.2 and estimate (2.16), we obtain the following result.

Lemma 2.4. *For any $\alpha_0 \in \mathbb{R}$ and any $\nu > -\frac{1}{2}$ there exists $T = T(\alpha_0, \nu) > 0$ such that for $t \geq T$ the following holds.*

(i) *The profile $\chi_{in}^{ap}(t)$ verifies*

$$(2.17) \quad \|\chi_{in}^{ap}\|_{L^\infty(0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|)t^{-\frac{1}{2}-\nu},$$

$$(2.18) \quad \|\rho^{-k} \partial_\rho^l \chi_{in}^{ap}\|_{L^\infty(0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu}, \quad 1 \leq k+l \leq 2,$$

$$(2.19) \quad \|\rho^{-k} \partial_\rho^l \chi_{in}^{ap}\|_{L^2(\rho^2 d\rho, 0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|)t^{-(\frac{1}{2}+\nu)(k+l-\frac{1}{2})}, \quad k+l \leq 2.$$

(ii) *The error $\mathcal{R}_{in}(t)$ admits the estimate*

$$(2.20) \quad \|\rho^{-k} \partial_\rho^l \mathcal{R}_{in}(t)\|_{L^2(\rho^2 d\rho, 0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq t^{-3(1+2\nu)/4-\epsilon_1(2N+1/2)}, \quad k+l \leq 2.$$

2.2. The self-similar region. We next consider the self-similar region $\frac{1}{10}t^{-\epsilon_1} \leq |x|t^{-1/2} \leq 10t^{\epsilon_2}$, where $0 < \epsilon_2 < 1/2$ to be fixed later. Write $\psi(x, t) = e^{i\alpha_0 \ln t} t^{-1/4} w(y, t)$, $y = t^{-1/2}|x|$. Then, $w(t)$ solves

$$(2.21) \quad itw_t = (\mathcal{L} + \alpha_0)w - |w|^4 w,$$

where $\mathcal{L} = -\Delta + \frac{i}{2}(\frac{1}{2} + y\partial_y)$.

Note that in the limit $\rho \rightarrow +\infty$, $y \rightarrow 0$ one has, at least, formally

$$(2.22) \quad t^{\nu/2} (W(\rho) + \sum_{k \geq 1} t^{-k(1+2\nu)} \chi_k(\rho)) = t^{-1/4} \sum_{n \geq 0} \sum_{0 \leq l \leq \frac{n}{2}} t^{-\frac{1}{4}(2n+1)(1+2\nu)} (\ln y + (\frac{1}{2} + \nu) \ln t)^l \sum_{k \geq l} \alpha_{l, 2k-n-1}^{(k)} y^{2k-n-1},$$

where $\alpha_{l,j}^{(k)}$, $k \neq 0$, are given by Lemma 2.2 and $\alpha_{l,j}^{(0)}$ come from the expansion of $W(\rho)$ as $\rho \rightarrow \infty$:

$$W(\rho) = \sum_{j \leq 0} \alpha_{0,j}^{(0)} \rho^j, \quad \alpha_{0,2m}^{(0)} = 0 \quad \forall m \in \mathbb{Z}.$$

¹This choice has no specific meaning here. To produce an approximate solution with an error verifying (2.6) it is sufficient to require $(2N+3)\epsilon_1 > 3(1+2\nu)/2$, $0 < \epsilon_1 < \frac{1+2\nu}{20}$, see (2.20) and (2.41), (2.42).

Eq. (2.22) suggests the following ansatz for w :

$$(2.23) \quad w(y, t) = \sum_{n \geq 0} \sum_{0 \leq l \leq \frac{n}{2}} t^{-\frac{1}{4}(2n+1)(1+2\nu)} (\ln y + (\frac{1}{2} + \nu) \ln t)^l A_{n,l}(y).$$

As it will become clear later, to prove Proposition 2.1, it is sufficient to consider only three first terms of expansion (2.23). Therefore, we look for an approximate solution of the form

$$\begin{aligned} w_{ss}^{ap}(y, t) = & t^{-(1+2\nu)/4} A_{0,0}(y) + t^{-3(1+2\nu)/4} A_{1,0}(y) \\ & + t^{-5(1+2\nu)/4} (A_{2,0}(y) + (\ln y + (\frac{1}{2} + \nu) \ln t) A_{2,1}(y)). \end{aligned}$$

Substituting this ansatz into the expression $-itw_t + (\mathcal{L} + \alpha_0)w - |w|^4 w$ one gets

$$(2.24) \quad \begin{aligned} -it\partial_t w_{ss}^{ap} + (\mathcal{L} + \alpha_0)w_{ss}^{ap} - |w_{ss}^{ap}|^4 w_{ss}^{ap} = & t^{-(1+2\nu)/4} S_{0,0}(y) + t^{-3(1+2\nu)/4} S_{1,0}(y) \\ & + t^{-5(1+2\nu)/4} (S_{0,0}(y) + (\ln y + (\frac{1}{2} + \nu) \ln t) S_{2,1}(y)) + S(y, t), \end{aligned}$$

where

$$\begin{aligned} S_{n,0}(y) &= (\mathcal{L} + \mu_n) A_{n,0}(y), \quad n = 0, 1, \\ S_{2,1}(y) &= (\mathcal{L} + \mu_2) A_{2,1}(y), \\ S_{2,0}(y) &= (\mathcal{L} + \mu_2) A_{2,0}(y) - i\nu A_{2,1}(y) - \frac{2}{y} \partial_y A_{2,1}(y) - \frac{A_{2,1}(y)}{y^2} - |A_{0,0}(y)|^4 A_{0,0}(y), \\ S(y, t) &= -|w_{ss}^{ap}(y, t)|^4 w_{ss}^{ap}(y, t) + t^{-5(1+2\nu)/4} |A_{0,0}(y)|^4 A_{0,0}(y). \end{aligned}$$

Here $\mu_n = \alpha_0 + \frac{i}{4}(2n+1)(1+2\nu)$.

We require that $S_{n,l} = 0$, $n = 0, 1, 2$, $l = 0, 1$, which means that the corresponding $A_{n,l}$ have to solve

$$(2.25) \quad \begin{cases} (\mathcal{L} + \mu_n) A_{n,0} = 0, & n = 0, 1, \\ (\mathcal{L} + \mu_2) A_{2,1} = 0, \\ (\mathcal{L} + \mu_2) A_{2,0} = i\nu A_{2,1} + \frac{2}{y} \partial_y A_{2,1} + \frac{A_{2,1}}{y^2} + |A_{0,0}|^4 A_{0,0} \end{cases}.$$

In addition, in order to have the matching with the inner region, $A_{n,l}$ have to satisfy

$$(2.26) \quad A_{n,l}(y) = \sum_{k \geq l} \alpha_{l,2k-n-1}^{(k)} y^{2k-n-1}, \quad y \rightarrow 0.$$

Lemma 2.5. *There exists a unique solution of (2.25) that as $y \rightarrow 0$ admits an asymptotic expansion of the form*

$$(2.27) \quad A_{n,l}(y) = \sum_{k \geq l} d_{n,k,l} y^{2k-n-1},$$

with $d_{0,0,0} = \alpha_{0,-1}^{(0)}$, $d_{1,1,0} = \alpha_{0,0}^{(1)}$ and $d_{2,1,0} = \alpha_{0,-1}^{(1)}$.

Proof. First of all note that the equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $e_1(y, \mu)$, $e_2(y, \mu)$ such that:

- (i) $e_1(y, \mu) = \frac{1}{y} + (\mu - \frac{i}{4})\tilde{e}_1(y, \mu)$, where \tilde{e}_1 is an entire function of y and μ , odd with respect to y ;
- (ii) e_2 is an entire function of y and μ , even with respect to y , and as $y \rightarrow 0$, $e_2(y, \mu) = 1 + O(y^2)$.

Two first equations of (2.25) together with (2.27) give

$$(2.28) \quad A_{0,0}(y) = \alpha_{0,-1}^{(0)} e_1(y, \mu_0), \quad A_{1,0}(y) = \alpha_{0,0}^{(1)} e_2(y, \mu_1).$$

We next consider the remaining equations of (2.25). Equation $(\mathcal{L} + \mu_2)A_{2,1}(y) = 0$ and (2.27) yield $A_{2,1}(y) = c_0 e_1(y, \mu_2)$, with some constant c_0 . Then, for $A_{2,0}$ we have $(\mathcal{L} + \mu_2)A_{2,0} = F$, where

$$F = c_0(i\nu + \frac{2}{y}\partial_y + y^{-2})e_1(y, \mu_2) + |A_{0,0}|^4 A_{0,0}.$$

As $y \rightarrow 0$, F has an asymptotic expansion of the form

$$F(y) = \sum_{i \geq -2} \kappa_i y^{2i-1},$$

with some coefficients κ_i , κ_{-2} and $\kappa_{-1} + c_0$ being independent of c_0 .

Write $A_{2,0}(y) = -\frac{\kappa_{-2}}{6y^3} + \tilde{A}_{2,0}(y)$. Then $\tilde{A}_{2,0}$ solves

$$(2.29) \quad (\mathcal{L} + \mu_2)\tilde{A}_{2,0} = \tilde{F},$$

where $\tilde{F} = F + \frac{\kappa_{-2}}{6}(\mathcal{L} + \mu_2)\frac{1}{y^3}$ has the following asymptotics as $y \rightarrow 0$:

$$\tilde{F}(y) = \sum_{i \geq -1} \tilde{\kappa}_i y^{2i-1}, \quad \tilde{\kappa}_{-1} = \tilde{\kappa}_{-1}^0 - c_0,$$

with $\tilde{\kappa}_{-1}^0$ independent of c_0 . Take $c_0 = \tilde{\kappa}_{-1}^0$. Then Eq. (2.29) has a unique solution of the form

$$\tilde{A}_{2,0}(y) = \alpha_{0,-1}^{(1)} e_1(y, \mu_2) + \text{a } C^\infty \text{ odd function.}$$

□

Remark 2.6. By uniqueness, $A_{n,l}$ given by Lemma 2.5 verify matching conditions (2.26). Note also that all $A_{n,l}$ are entire functions of α_0 and ν .

We next study the behavior of $A_{n,l}$ as $y \rightarrow +\infty$. To this purpose notice that for any $\mu \in \mathbb{C}$, equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $f_1(y, \mu)$, $f_2(y, \mu)$ such that yf_1 , yf_2 are smooth functions in both variables and as $y \rightarrow +\infty$ one has

$$(2.30) \quad f_1(y, \mu) = y^{-1/2+2i\mu}(1 + O(y^{-2})), \quad f_2(y, \mu) = e^{i\frac{y^2}{4}} y^{-5/2-2i\mu}(1 + O(y^{-2})).$$

These asymptotics are uniform in μ on compact subsets of \mathbb{C} and can be differentiated any number of times with respect to y .

Decomposing $A_{1,0}$, $A_{2,0}$, $A_{2,1}$ in the basis f_1 , f_2 one gets

$$(2.31) \quad \begin{aligned} A_{n,0}(y) &= d_1^n f_1(y, \mu_n) + d_2^n f_2(y, \mu_n), \quad n = 0, 1, \\ A_{2,1}(y) &= d_1^2 f_1(y, \mu_2) + d_2^2 f_2(y, \mu_2), \end{aligned}$$

with some coefficients d_j^n , $j = 1, 2$, $n = 0, 1, 2$. As a consequence, as $y \rightarrow +\infty$, one has

$$(2.32) \quad \begin{aligned} A_{0,0}(y) &= d_1^0 y^{2i\alpha_0-1-\nu} (1 + O(y^{-2})) + d_2^0 e^{iy^2/4} y^{-2i\alpha_0-2+\nu} (1 + O(y^{-2})), \\ A_{1,0}(y) &= d_1^1 y^{2i\alpha_0-2-3\nu} (1 + O(y^{-2})) + d_2^1 e^{iy^2/4} y^{-2i\alpha_0-1+3\nu} (1 + O(y^{-2})), \\ A_{2,1}(y) &= d_1^2 y^{2i\alpha_0-3-5\nu} (1 + O(y^{-2})) + d_2^2 e^{iy^2/4} y^{-2i\alpha_0+5\nu} (1 + O(y^{-2})). \end{aligned}$$

Asymptotics (2.32) can be differentiated any number of times with respect to y .

Let us now consider $A_{2,0}$ and write it as

$$(2.33) \quad A_{2,0}(y) = 2d_1^2 \nu \ln y f_1(y, \mu_2) - 2(\nu + 1)d_2^2 \ln y f_2(y, \mu_2) + \hat{A}_{2,0}(y).$$

Then $\hat{A}_{2,0}(y)$ solves

$$(2.34) \quad (\mathcal{L} + \mu_2) \hat{A}_{2,0} = G,$$

with $G = d_2^2 G_1 + G_2$, where

$$\begin{aligned} G_1 &= -d_2^2 (1 + 2\nu) (2y^{-1} \partial_y + y^{-2} - i) f_2(y, \mu_2), \\ G_2 &= |A_{0,0}|^4 A_{0,0} + d_1^2 (1 + 2\nu) (2y^{-1} \partial_y + y^{-2}) f_1(y, \mu_2). \end{aligned}$$

It follows from the asymptotics (2.30), (2.32) that G_j , $j = 1, 2$, has the following behavior as $y \rightarrow +\infty$,

$$\begin{aligned} G_1(y) &= e^{iy^2/4} y^{-2i\alpha_0} G_{1,1}(y), \quad G_2(y) = \sum_{m=-2}^3 e^{imy^2/4} y^{-2i\alpha_0\nu(2m-1)} G_{2,m}(y), \\ \partial_y^l G_{1,1}(y) &= O(y^{-2+5\nu-l}), \\ \partial_y^l G_{2,m}(y) &= O(y^{-5-5\nu-|m|(1-2\nu)-l}), \quad -2 \leq m \leq 3, \end{aligned}$$

for any $l \geq 0$, provided ν is sufficiently small.

Integrating (2.34) one gets

$$(2.35) \quad \hat{A}_{2,0}(y) = \lambda_1 f_1(y, \mu_2) + \lambda_2 f_2(y, \mu_2) + d_2^2 g_1(y) + g_2(y).$$

Here λ_i , $i = 1, 2$, is a constant and g_i , $i = 1, 2$, is the solution of $(\mathcal{L} + \mu_2)g_i = G_i$, with the following behavior as $y \rightarrow +\infty$:

$$(2.36) \quad \begin{aligned} g_1(y) &= e^{iy^2/4} y^{-2i\alpha_0} g_{1,1}(y), \\ g_2(y) &= \sum_{m=-2}^3 e^{imy^2/4} y^{-2i\alpha_0\nu(2m-1)} g_{2,m}(y), \\ \partial_y^l g_{1,1}(y) &= O(y^{-2+5\nu-l}), \\ \partial_y^l g_{2,m}(y) &= O(y^{-5-5\nu-m(1-2\nu)-l}), \quad m = 0, 1 \\ \partial_y^l g_{2,m}(y) &= O(y^{-7-5\nu-|m|(1-2\nu)-l}), \quad m = -2, -1, 2, 3, \end{aligned}$$

for any $l \geq 0$.

Denote

$$\begin{aligned} u_{ss}^{ap}(\rho, t) &= t^{-(1+2\nu)/4} w_{ss}^{ap}(t^{-(1+2\nu)/2} \rho, t), \\ \chi_{ss}^{ap}(\rho, t) &= u_{ss}^{ap}(\rho, t) - W(\rho), \\ \mathcal{R}_{ss}(\rho, t) &= t^{-5(1+2\nu)/4} S(t^{-(1+2\nu)/2} \rho, t). \end{aligned}$$

The next lemma is a direct consequence of (2.26), (2.30), (2.32), (2.33), (2.35) and (2.36).

Lemma 2.7. *For any $\alpha_0, \nu \in \mathbb{R}$ sufficiently small there exists $T(\alpha_0, \nu) > 0$ such that for $t \geq T(\alpha_0, \nu)$ the following holds.*

(i) $\chi_{ss}^{ap}(t)$ verifies

$$(2.37) \quad \|\chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-\frac{1}{2}-\nu},$$

$$(2.38) \quad \|\rho^{-k} \partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-1-2\nu}, \quad k+l=1,$$

$$(2.39) \quad \|\rho^{-k} \partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq C(|\alpha_0| + |\nu|)t^{-1-2\nu}, \quad k+l=2,$$

$$(2.40) \quad \|\rho^{-k} \partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^2(\rho^2 d\rho, \frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-(1+2\nu)(1-2\epsilon_2)/4}, \quad 1 \leq k+l \leq 2,$$

(ii) The error $\mathcal{R}_{ss}(t)$ admits the estimate

$$(2.41) \quad \|\rho^{-k} \partial_\rho^l \mathcal{R}_{ss}(t)\|_{L^2(\rho^2 d\rho, \frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-(2+\frac{1}{4})(1+2\nu)+5\epsilon_1/2}, \quad 0 \leq k+l \leq 2.$$

(iii) The difference $u_{in}^{ap}(\rho, t) - u_{ss}^{ap}(\rho, t)$ verifies

$$(2.42) \quad |\partial_\rho^l (u_{in}^{ap}(t) - u_{ss}^{ap}(t))| \leq C\rho^{-2-l}t^{-(1+2\nu)}(\ln t + t^{3(1+2\nu)/2-(2N+3)\epsilon_1}),$$

for any $l \geq 0$ and $\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1}$.

2.3. The remote region. We next consider the remote region $|x| \geq \frac{1}{10}t^{1/2+\epsilon_2}$. In this region we take as an approximate solution to (1.1) the following radial profile:

$$\psi_{out}^{ap}(x, t) = v_1(x, t) + v_2(x, t) + v_3(x, t),$$

where

$$\begin{aligned} v_1(x, t) &= e^{i\alpha_0 \ln t} [d_1^0 t^{-(1+\nu)/2} f_1(y, \mu_0) + d_1^1 t^{-(2+3\nu)/2} f_1(y, \mu_1)], \quad y = t^{-1/2}|x|, \\ v_2(x, t) &= \Theta_\delta\left(\frac{x}{t}\right) e^{i\alpha_0 \ln t} [d_2^0 t^{-(1+\nu)/2} f_2(y, \mu_0) + d_2^1 t^{-(2+3\nu)/2} f_2(y, \mu_1) + \\ &\quad + t^{-(3+5\nu)/2} (d_2^2 g_1(y) - (d_2^2(2\nu+1) \ln\left(\frac{|x|}{t}\right) - \lambda_2) f_2(y, \mu_2))], \end{aligned}$$

$$\Theta_\delta(\xi) = \Theta\left(\frac{\xi}{\delta}\right), \quad \Theta \in C_0^\infty(\mathbb{R}^3) \text{ is radial, } \Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}.$$

Finally, $v_3(x, t)$ is given by

$$v_3(x, t) = \frac{e^{i\frac{|x|^2}{4t}}}{t^{5/2}} \hat{v}_3\left(\frac{x}{t}\right), \quad \hat{v}_3 = -iz\Delta\Theta_\delta - 2i\nabla z \cdot \nabla\Theta_\delta,$$

where

$$z(\xi) = d_2^0 |\xi|^{-2i\alpha_0-2+\nu} + d_2^1 |\xi|^{-2i\alpha_0-1+3\nu} - (d_2^2(2\nu+1) \ln |\xi| - \lambda_2) |\xi|^{-2i\alpha_0+5\nu}.$$

It follows from the asymptotics (2.30) that for $t \geq T$ with some $T = T(\delta) > 0$ and any $l \geq 0$, one has

$$(2.43) \quad \begin{aligned} |\nabla^l v_1(x, t)| &\leq C_l |x|^{-l-1-\nu}, \quad \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x|, \\ |\nabla^l v_2(x, t)| &\leq \frac{C_l}{t^{3/2}} \left| \frac{x}{t} \right|^{l-2+\nu}, \quad \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 2\delta t. \end{aligned}$$

Furthermore, v_2 can be written as

$$(2.44) \quad \begin{aligned} v_2(x, t) &= v_{2,0}(x, t) + v_{2,1}(x, t), \\ v_{2,0}(x, t) &= \frac{e^{i\frac{|x|^2}{4t}}}{t^{3/2}} \Theta_\delta \left(\frac{x}{t} \right) z \left(\frac{x}{t} \right), \quad v_{2,1}(x, t) = \frac{e^{i\frac{|x|^2}{4t}}}{t^{3/2}} \Theta_\delta \left(\frac{x}{t} \right) \hat{v}_{2,1}(x, t), \end{aligned}$$

with $\hat{v}_{2,1}$ verifying, for any $l \geq 0$,

$$(2.45) \quad |\nabla^l \hat{v}_{2,1}(x, t)| \leq C_l t^{3-\nu} |x|^{-l-4+\nu}, \quad \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 2\delta t.$$

We next address v_3 . One has

$$(2.46) \quad \begin{aligned} \|\nabla^l v_3(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C_l t^{-5/2} \delta^{-4+l+\nu}, \\ \|\nabla^l v_3(t)\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C_l t^{-1} \delta^{-5/2+l+\nu}, \end{aligned}$$

for any $l \geq 0$ and $t \geq T(\delta)$.

As a direct consequence of estimates (2.43), (2.45), (2.46), one obtains

$$(2.47) \quad \begin{aligned} \|\psi_{out}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C t^{-(\frac{1}{2}+\varepsilon_2)(1+\nu)}, \\ \| |x|^{-1} \psi_{out}^{ap}(t) \|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C t^{-5/4}, \\ \|\nabla^l \psi_{out}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C t^{-5/4}, \quad l = 1, 2, \\ \|\nabla^l \psi_{out}^{ap}(t)\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C \delta^{\nu+l-1/2}, \quad l = 1, 2, \\ \|\nabla^l (\psi_{out}^{ap}(t) - v_{2,0}(t))\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C t^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)}, \quad l = 1, 2, \\ \| |x|^{-1} (\psi_{out}^{ap}(t) - v_{2,0}(t)) \|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} &\leq C t^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)}, \end{aligned}$$

provided $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$, ν is sufficiently small and $t \geq T(\delta)$.

Denote

$$\psi_{ss}^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{-1/4} w_{ss}^{ap}(t^{-1/2}|x|, t),$$

and consider the difference $\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t)$. For $\frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 10 t^{1/2+\varepsilon_2}$ one has

$$(2.48) \quad \psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{-(3+5\nu)/2} ((d_1^2(1+2\nu) \ln |x| + \lambda_1) f_1(y, \mu_2) + g_2(y)),$$

which together with (2.30) and (2.36) implies that

$$(2.49) \quad |\nabla^l(\psi_{out}^{ap} - \psi_{ss}^{ap})| \leq C_l(|\ln t|t^{-(\frac{1}{2}+\varepsilon_2)(3+5\nu+l)} + t^{-(\frac{1}{2}+\varepsilon_2)(3+5\nu+1)}),$$

for any $l \geq 0$ and $\frac{1}{10}t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2}$, provided ν is sufficiently small.

We next analyze the error $R_{out}(t) = -i\partial_t \psi_{out}^{ap}(t) - \Delta \psi_{out}^{ap}(t) - |\psi_{out}^{ap}(t)|^4 \psi_{out}^{ap}(t)$. It has the form

$$(2.50) \quad R_{out}(x, t) = -\frac{e^{i\frac{|x|^2}{4t}}}{t^{9/2}} \left[t\hat{v}_{2,1}(x, t)\Delta\Theta_\delta\left(\frac{x}{t}\right) + 2t^2\nabla\hat{v}_{2,1}(x, t) \cdot \nabla\Theta_\delta\left(\frac{x}{t}\right) + \Delta\hat{v}_3\left(\frac{x}{t}\right) \right] - |\psi_{out}^{ap}|^4 \psi_{out}^{ap}.$$

Combined with (2.43), (2.45), (2.46), representation (2.50) gives for $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and ν sufficiently small,

$$(2.51) \quad \|\nabla^l R_{out}(t)\|_{L^2(|x| \geq \frac{1}{10}t^{1/2+\varepsilon_2})} \leq Ct^{-\frac{9}{4}(1+2\nu)}, \quad t \geq T(\delta), \quad l = 0, 1, 2.$$

2.4. Proof of Proposition 2.1. We are now in position to conclude the proof of Prop. 2.1. Fix ε_2 such that $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and consider the radial profile $\psi^{ap}(x, t)$ defined by

$$\begin{aligned} \psi^{ap}(x, t) = & \Theta(t^{-1/2+\varepsilon_1}x)\psi_{in}^{ap}(x, t) + (1 - \Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)\psi_{ss}^{ap}(x, t) \\ & + (1 - \Theta(t^{-1/2-\varepsilon_2}x))\psi_{out}^{ap}(x, t), \quad x \in \mathbb{R}^3, \end{aligned}$$

where $\psi_{in}^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{\nu/2} u_{in}^{ap}(t^\nu |x|, t)$. Write ψ^{ap} as $\psi^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{\nu/2} (W(y) + \chi^{ap}(y, t))$, $y = t^\nu x$. By Lemma 2.4 (estimates (2.17), (2.18)), Lemma 2.7 (estimates (2.37), (2.38), (2.39)) and (2.47) one has

$$(2.52) \quad \|\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-(1+2\nu)/2}$$

$$(2.53) \quad \| |y|^{-1} \chi^{ap}(t) \|_{L^\infty} + \|\nabla \chi^{ap}(t)\|_{L^\infty} \leq Ct^{-1-2\nu},$$

$$(2.54) \quad \| |y|^{-2} \chi^{ap}(t) \|_{L^\infty} + \| |y|^{-1} \nabla_y \chi^{ap}(t) \|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu},$$

$$(2.55) \quad \|\nabla^2 \chi^{ap}(t)\|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu}.$$

All the estimates stated in this subsection are valid for ν sufficiently small and $t \geq T(\alpha_0, \nu, \delta)$.

Futhermore, it follows from Lemma 2.4 (estimate (2.19)), Lemma 2.7 (estimate (2.39)) and two last inequalities in (2.47) that

$$(2.56) \quad \begin{aligned} \|\nabla^l \chi^{ap}(t)\|_{L^2(|y| \leq 10t^{1/2+\nu+\varepsilon_2})} & \leq Ct^{-(1+2\nu)(1-2\varepsilon_2)/4}, \quad l = 1, 2, \\ \|\nabla^l(\chi^{ap}(t) - \chi_0^{ap}(t))\|_{L^2(|y| \geq t^{1/2+\nu+\varepsilon_2})} & \leq Ct^{-(1+2\nu)/4}, \quad l = 1, 2, \end{aligned}$$

where $\chi_0^{ap}(y, t) = e^{-i\alpha_0 \ln t} t^{-\nu/2} v_{2,0}(t^{-\nu} y, t)$.

Inequalities (2.56) imply, in particular,

$$\|\nabla^l \chi^{ap}(t)\|_{L^2(\mathbb{R}^3)} \leq Ct^{-\nu(l-1)} \delta^{\nu+l-1/2}, \quad l = 1, 2.$$

Moreover, introducing $\zeta^*(x) = \pi^{-3/2} e^{3i\pi/4} \int_{\mathbb{R}^3} d\xi e^{ix \cdot \xi} \Theta_\delta(2\xi) z(2\xi)$ and observing that $\zeta^* \in \dot{H}^s(\mathbb{R}^3)$ for any $s > 1/2 - \nu$, and $\|\nabla^l(v_{2,0} - e^{i\Delta t} \zeta^*)\|_{L^2(|x| \geq t^\gamma)} \rightarrow 0$ as $t \rightarrow +\infty$ for any $\gamma > \frac{1-2\nu}{3-2\nu}$ and any $l \geq 1$, one obtains that

$$e^{i\alpha(t)} \lambda^{1/2}(t) \chi^{ap}(\lambda(t) \cdot, t) - e^{it\Delta} \zeta^* \rightarrow 0 \text{ in } \dot{H}^1 \cap \dot{H}^2 \text{ as } t \rightarrow +\infty.$$

This concludes the proof of the first part of Prop. 2.1.

We next consider the error $R = -i\psi_t^{ap} - \Delta\psi^{ap} - |\psi^{ap}|^4\psi^{ap}$. It has the form

$$R = E_1 + E_2 + E_3 + E_4.$$

where

$$\begin{aligned} E_1 &= i\left(\frac{1}{2} - \varepsilon_1\right)t^{-1}(\psi_{in}^{ap}(x, t) - \psi_{ss}^{ap}(x, t))\tilde{\Theta}(t^{-1/2+\varepsilon_1}x) \\ &\quad - 2t^{-1/2+\varepsilon_1}(\nabla\psi_{in}^{ap}(x, t) - \nabla\psi_{ss}^{ap}(x, t)) \cdot \nabla\Theta(t^{-1/2+\varepsilon_1}x) \\ &\quad - t^{-1+2\varepsilon_1}(\psi_{in}^{ap}(x, t) - \psi_{ss}^{ap}(x, t))\Delta\Theta(t^{-1/2+\varepsilon_1}x), \\ E_2 &= i\left(\frac{1}{2} + \varepsilon_2\right)t^{-1}(\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t))\tilde{\Theta}(t^{-1/2-\varepsilon_2}x) \\ &\quad - 2t^{-1/2-\varepsilon_2}(\nabla\psi_{ss}^{ap}(x, t) - \nabla\psi_{out}^{ap}(x, t)) \cdot \nabla\Theta(t^{-1/2-\varepsilon_2}x) \\ &\quad - t^{-1-2\varepsilon_2}(\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t))\Delta\Theta(t^{-1/2-\varepsilon_2}x), \\ \tilde{\Theta}(\xi) &= \xi \cdot \nabla\Theta(\xi), \end{aligned}$$

and E_3, E_4 are given by

$$\begin{aligned} E_3 &= \Theta(t^{-1/2+\varepsilon_1}x)R_{in}(x, t) + (1 - \Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)R_{ss}(x, t) \\ &\quad + (1 - \Theta(t^{-1/2-\varepsilon_2}x))R_{out}(x, t), \\ E_4 &= \Theta(t^{-1/2+\varepsilon_1}x)(|\psi_{in}^{ap}|^4\psi_{in}^{ap} - |\psi^{ap}|^4\psi^{ap}) \\ &\quad + (1 - \Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)(|\psi_{ss}^{ap}|^4\psi_{ss}^{ap} - |\psi^{ap}|^4\psi^{ap}) \\ &\quad + (1 - \Theta(t^{-1/2-\varepsilon_2}x))(|\psi_{out}^{ap}|^4\psi_{out}^{ap} - |\psi^{ap}|^4\psi^{ap}). \end{aligned}$$

Here

$$R_{in}(x, t) = e^{i\alpha_0 \ln t} t^{5\nu/2} \mathcal{R}_{in}(t^\nu|x|, t), \quad R_{ss}(x, t) = e^{i\alpha_0 \ln t} t^{5\nu/2} \mathcal{R}_{ss}(t^\nu|x|, t).$$

First we adress E_1 . By Lemma 2.7 (iii) we have

$$(2.57) \quad \|E_1\|_{H^2} \leq Ct^{-9(1+2\nu)/4+\nu+5\varepsilon_1/2} \ln t \leq Ct^{-(2+\frac{3}{20})(1+2\nu)}.$$

Similarly, from (2.49) we get for E_2 :

$$(2.58) \quad \|E_2\|_{H^2} \leq Ct^{-1-(\frac{1}{2}+\varepsilon_2)(\frac{3}{2}+5\nu)} \ln t \leq Ct^{-(2+\frac{1}{4})(1+2\nu)}.$$

Next, we consider E_3 . From Lemma 2.4 (ii), Lemma 2.7 (ii) and (2.51) it is apparent that

$$(2.59) \quad \|E_3\|_{H^2} \leq Ct^{-\frac{9}{4}(1+2\nu)+5\varepsilon_1/2} \leq Ct^{-(2+\frac{3}{20})(1+2\nu)}.$$

Finally, applying Lemma 2.4 (estimates (2.17), (2.18)), Lemma 2.7 (estimates (2.37), (2.38), (2.39), (2.42)) and (2.47), (2.49), it is not difficult to check that

$$(2.60) \quad \|E_4\|_{H^2} \leq Ct^{-3(1+2\nu)}.$$

Combining (2.57), (2.58), (2.59), (2.60), we get (2.6), which concludes the proof of Prop. 2.1.

3. CONSTRUCTION OF AN EXACT SOLUTION

We are now in position to prove Theorem 1.1. Consider (1.1) and write $\psi(x, t) = e^{i\alpha_0 \ln t} t^{\nu/2} U(y, \tau)$, where $y = t^\nu x$ and $\tau = \frac{t^{1+2\nu}}{1+2\nu}$. Further decomposing U as

$$U(y, \tau) = U^{ap}(y, \tau) + f(y, \tau), \quad U^{ap}(y, \tau) = e^{-i\alpha_0 \ln t} t^{-\nu/2} \psi^{ap}(x, t),$$

where ψ^{ap} is the approximate solution of (1.1) given by Prop. (2.1), we get the following equation for the remainder f

$$(3.1) \quad i\vec{f}_\tau = \mathcal{H}(\tau)\vec{f} + \mathcal{F}(f) + r, \quad \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix},$$

where

$$\mathcal{H}(\tau) = H + \tau^{-1}l,$$

$$H = -\Delta\sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad l = \frac{\alpha_0}{2\nu+1}\sigma_3 - i\frac{\nu}{2\nu+1}\left(\frac{1}{2} + y \cdot \nabla\right),$$

$$\mathcal{F}(f) = \begin{pmatrix} F(f) \\ -\overline{F(f)} \end{pmatrix}, \quad F(f) = F_1(f) + F_2(f)$$

$$F_1(f) = \mathcal{V}_1(\tau)f + \mathcal{V}_2(\tau)\bar{f},$$

$$\mathcal{V}_1(\tau) = 3(W^4 - |U^{ap}(\tau)|^4), \quad \mathcal{V}_2(\tau) = 2(W^4 - (U^{ap}(\tau))^2|U^{ap}(\tau)|^2),$$

$$F_2(f) = -|U^{ap} + f|^4(U^{ap} + f) + |U^{ap}|^4U^{ap} + 3|U^{ap}|^4f + 2(U^{ap})^2|U^{ap}|^2\bar{f},$$

$$r = \begin{pmatrix} \mathbf{r} \\ -\bar{\mathbf{r}} \end{pmatrix}, \quad \mathbf{r}(y, \tau) = t^{-5\nu/2} e^{-i\alpha_0 \ln t} R(x, t).$$

R being the error given by Prop. 2.1. Note that by Prop. 2.1 one has

$$(3.2) \quad \|\mathcal{V}_i(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(|\alpha_0| + |\nu|)\tau^{-1}, \quad i = 1, 2,$$

$$(3.3) \quad \|U^{ap}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C,$$

$$(3.4) \quad \|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2-\frac{1}{8}},$$

for any $\tau \geq \tau_0$ with some $\tau_0 > 0$.

Our intention is to solve (3.1) with zero condition at $\tau = +\infty$ by a fix point argument. To carry out this analysis we will need some energy type estimates for the linearized equation $i\vec{f}_\tau = \mathcal{H}(\tau)\vec{f}$. The required estimates are collected in the next subsection, their proofs being removed to Section 4.

3.1. Linear estimates. We start by recalling some basic spectral properties of the operator H (a more detailed discussion and the proofs can be found, for example, in [5]). Since we are considering only radial solutions, we will view H as an operator on $L^2_{rad}(\mathbb{R}^3; \mathbb{C}^2)$ with domain $D(H) = H^2_{rad}(\mathbb{R}^3; \mathbb{C}^3)$. H satisfies the relations

$$\sigma_3 H \sigma_3 = H^*, \quad \sigma_1 H \sigma_1 = -H.$$

The essential spectrum of H fills up the real axis. The discrete spectrum of H consists of two simple purely imaginary eigenvalues $i\lambda_0, -i\lambda_0$, $\lambda_0 > 0$. The corresponding eigenfunctions ζ_+, ζ_- are in $\mathcal{S}(\mathbb{R}^3)$ and can be chosen in such a way that $\zeta_- = \sigma_1 \zeta_+ = \bar{\zeta}_+$. Notice also that $HW\begin{pmatrix} 1 \\ -1 \end{pmatrix} = HW_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$, which means that H has a resonance at zero.

Consider the projection of the linearized equation $i\vec{f}_\tau = \mathcal{H}(\tau)\vec{f}$ onto the essential spectrum of H :

$$(3.5) \quad i\vec{f}_\tau = P\mathcal{H}(\tau)P\vec{f}.$$

Here P is the spectral projection of H onto the essential spectrum given by

$$P = I - P_+ - P_-, \quad P_\pm = \frac{\langle \cdot, \sigma_3 \zeta_\mp \rangle}{\langle \zeta_\pm, \sigma_3 \zeta_\mp \rangle} \zeta_\pm,$$

$\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^3; \mathbb{C}^2)$.

Let $U(\tau, s)$ be the propagator associated to Eq. (3.5). In Section 4 we prove the following results.

Proposition 3.1. *There exists a constant $C > 0$ such that*

$$\|U(\tau, s)f\|_{H^2} \leq C \left(\frac{s}{\tau}\right)^{C(|\alpha_1| + |\nu_1|)} \|f\|_{H^2},$$

for any $s \geq \tau > 0$ and any $f \in H^2_{rad}$. Here $\alpha_1 = \frac{\alpha_0}{1+2\nu}$, $\nu_1 = \frac{\nu}{1+2\nu}$.

3.2. Contraction argument. We now transforme (3.1) into a fix point problem. Rewrite (3.1) in the following integral form

$$(3.6) \quad f(\tau) = J(f)(\tau),$$

where

$$\begin{aligned}
J(f)(\tau) &= J_0(f)(\tau) + J_+(f)(\tau) + J_-(f)(\tau), \\
J_0(f)(\tau) &= i \int_{\tau}^{+\infty} ds U(\tau, s) P(\mathcal{F}_1(f(s)) + r(s)), \\
J_+(f)(\tau) &= i \int_{\tau}^{+\infty} ds e^{\lambda_0(\tau-s)} P_+(\mathcal{F}_2(f(s)) + r(s)), \\
J_-(f)(\tau) &= -i \int_{\tau_1}^{\tau} ds e^{-\lambda_0(\tau-s)} P_-(\mathcal{F}_2(f(s)) + r(s)), \\
\mathcal{F}_1(f) &= \mathcal{F}(f) + s^{-1} l(P_+ + P_-) \vec{f}, \\
\mathcal{F}_2(f) &= \mathcal{F}(f) + s^{-1} l \vec{f},
\end{aligned}$$

$\tau_1 \geq \max\{\tau_0, 1\}$ to be fixed later (slightly abusing notation we identify in (3.6) \mathbb{C}^2 vectors of the form $\begin{pmatrix} f \\ \vec{f} \end{pmatrix}$ with their first component f).

Our intention is to view J as a mapping in the space $C([\tau_1, +\infty), H_{rad}^2)$ equipped with the norm $|||f||| = \sup_{\tau \geq \tau_1} \|f(\tau)\|_{H^2} \tau^{1+1/16}$, and to show that J is contraction of the unite ball $|||f||| \leq 1$ into itself provided $|\alpha_0| + |\nu|$ is sufficiently small and τ_1 is chosen sufficiently large. Indeed, by (3.3), (3.2) one has, for any $f, g \in H^2$ with $\|f\|_{H^2} \leq 1, \|g\|_{H^2} \leq 1$,

$$\|\mathcal{F}_1(f) - \mathcal{F}_1(g)\|_{H^2} \leq C(\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|)\tau^{-1})\|f - g\|_{H^2},$$

$$\|P_{\pm}(\mathcal{F}_2(f) - \mathcal{F}_2(g))\| \leq C(\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|)\tau^{-1})\|f - g\|_{H^2},$$

which together with (3.4) and Prop. 3.1 gives

$$|||J(f)||| \leq \frac{1}{2} + C\tau_1^{-1/16}, \quad |||J(f) - J(g)||| \leq (\frac{1}{2} + C\tau_1^{-1/16})|||f - g|||,$$

for any $f, g \in \{|||h||| \leq 1\}$, provided $|\alpha_0| + |\nu|$ is sufficiently small. This means that for τ_1 sufficiently large, J is a contraction of the unit ball $|||f||| \leq 1$ into itself and consequently, has a unique fixe point f that satisfies

$$\|f(\tau)\|_{H^2} \leq \tau^{-1-1/16}, \quad \forall \tau \geq \tau_1,$$

which together with Prop. 2.1 gives Theorem 1.1.

4. LINEARIZED EVOLUTION

In this section we prove Prop. 3.1. The proof will be achieved by combining the results of [5] with a careful spectral analysis of the operator H around zero energy. The section organized as follows. In subsection 1 we consider the operator H as before, restricted to the subspace of radial functions, and construct a basis of Jost solutions for the equation $H\zeta = E\zeta$. In subsection 2 we study the spectral decomposition of H near $E = 0$. In subsection 3 we prove Prop. 3.1 by combining the results of the previous two subsections with the coercivity properties of H established in [5].

4.1. Solutions to the equation $H\zeta = E\zeta$. In this subsection we construct a basis of Jost solutions of the equation $H\zeta = E\zeta$, $E \in \mathbb{R}$. Since the subject is completely standard we will only briefly sketch the proofs (see also [1], [7] for a closely related construction in the context of energy subcritical NLS). Recall that

$$H = -(\partial_\rho^2 + 2\rho^{-1}\partial_\rho)\sigma_3 + V(\rho), \quad V = \begin{pmatrix} V_1 & V_2 \\ -V_2 & -V_1 \end{pmatrix},$$

$$V_1(\rho) = -3W^4(\rho), \quad V_2(\rho) = -2W^4(\rho), \quad W(\rho) = (1 + \rho^2/3)^{-1/2}.$$

We emphase that $V(\rho)$ is a smooth function of ρ that decays as ρ^{-4} as $\rho \rightarrow \infty$. Since $\sigma_1 H = -H\sigma_1$ it suffices to consider the case $E \geq 0$, so we write $E = k^2$, $k \geq 0$. It will be convenient for us to remove the first derivative in H . Set $f = \rho\zeta$, then one gets

$$(4.1) \quad \tilde{H}f = Ef, \quad \tilde{H} = -\partial_\rho^2\sigma_3 + V(\rho).$$

We will consider the operator \tilde{H} on \mathbb{R} , to recover the original radial \mathbb{R}^3 problem it suffices to restrict \tilde{H} to the subspace of odd functions.

We start by constructing the most rapidly decaying solution to (4.1).

Lemma 4.1. *For all $k \geq 0$ there exists a real solution $f_3(\rho, k)$ of the equation*

$$(4.2) \quad \tilde{H}f = k^2 f,$$

such that $f_3(\rho, k) = e^{-k\rho}\chi_3(\rho, k)$, where χ_3 is C^∞ function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}_+^$ verifying $\chi_3(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a(\rho, k)$,*

$$(4.3) \quad \begin{aligned} |\partial_\rho^l \partial_k^m a(\rho, k)| &\leq C_l \langle \rho \rangle^{-2-l+m} (1 + k \langle \rho \rangle)^{-1-m}, \quad m = 0, 1, \\ |\partial_\rho^l \partial_k^2 a(\rho, k)| &\leq C_l \langle \rho \rangle^{-l} (1 + k \langle \rho \rangle)^{-3} \ln \left(\frac{1}{k \langle \rho \rangle} + 2 \right), \end{aligned}$$

for all $\rho \geq 0$, $k > 0$ and $l \geq 0$.

Proof. One writes the following integral equation for χ_3

$$\chi_3(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_\rho^{+\infty} K(\rho - s, k) \sigma_3 V(s) \chi_3(s, k) ds,$$

$$K(\xi, k) = \begin{pmatrix} \frac{\sin k\xi}{k} & 0 \\ 0 & \frac{\sinh k\xi}{k} \end{pmatrix} e^{k\xi}.$$

The statement of the lemma follows then from the estimate

$$|\partial_k^l K(\xi, k)| \leq C_l \frac{|\xi|^{l+1}}{\langle k\xi \rangle^{l+1}}, \quad \xi \leq 0, \quad k \geq 0, \quad l \geq 0$$

and the decay properties of V :

$$|\partial_\rho^l V(\rho)| \leq C_l \langle \rho \rangle^{-4-l}, \quad \rho \in \mathbb{R}, \quad l \geq 0,$$

by standard Volterra iterations. □

We next construct the oscillating solutions to Eq. (4.2).

Lemma 4.2. *For all $k \geq 0$ there exists a solution $f_1(\rho, k)$ of Eq. (4.2) such that f_1 is a smooth function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}_+^*$ of the form $f_1(\rho, k) = e^{ik\rho}(\binom{1}{0} + b(\rho, k))$, where b verifies*

$$(4.4) \quad \begin{aligned} |b(\rho, k)| &\leq C(\langle \rho \rangle^{-2} + ke^{-k\rho}), \\ |\partial_\rho b(\rho, k)| &\leq C(\langle \rho \rangle^{-3} + k^2 e^{-k\rho}), \\ |\partial_k b(\rho, k)| &\leq C(\langle \rho \rangle^{-1} + \langle k\rho \rangle e^{-k\rho}), \\ |\partial_{\rho k}^2 b(\rho, k)| &\leq C(\langle \rho \rangle^{-2} + k \langle k\rho \rangle e^{-k\rho}), \end{aligned}$$

for all $\rho \geq 0$, $0 \leq k \lesssim 1$. In addition, one has

$$|\partial_k^2 b(\rho, k)| \leq C \ln \left(\frac{1}{k} + 1 \right),$$

for all $0 \leq \rho \lesssim 1$, $0 < k \lesssim 1$.

Proof. To construct f_1 we will reduce the order of the system (4.2) by means of the substitution $f_1 = z_0 f_3 + z_1 \binom{1}{0}$. Further setting $z_2 = z'_0 f_{3,2}$, $f_3 = \binom{f_{3,1}}{f_{3,2}}$, we get that $z = \binom{z_1}{z_2}$ solves

$$(4.5) \quad \begin{aligned} -z_1'' - k^2 z_1 + V_{11} z_1 + V_{12} z_2 &= 0, \\ -z_2' + k z_2 + V_{21} z_1 + V_{22} z_2 &= 0. \end{aligned}$$

Here

$$\begin{aligned} V_{11} &= V_1 - V_2 \frac{f_{3,1}}{f_{3,2}}, \quad V_{12} = \frac{2}{f_{3,2}^2} (f_{3,1} f'_{3,2} - f'_{3,1} f_{3,2}), \\ V_{21} &= V_2, \quad V_{22} = -\frac{1}{f_{3,2}} (f'_{3,2} + k f_{3,2}). \end{aligned}$$

By Lemma 4.1, there exists $R > 0$ independent of k , such that the functions $V_{ij}(\rho, k)$, $i, j = 1, 2$, are smooth in both variables for $k > 0$ and $\rho \geq R$ and verify for all $l \geq 0$, $\rho \geq R$, $k > 0$,

$$(4.6) \quad \begin{aligned} |\partial_\rho^l V_{j1}(\rho, k)| &\leq C_l \langle \rho \rangle^{-4-l}, \quad j = 1, 2, \\ |\partial_\rho^l \partial_k V_{11}(\rho, k)| &\leq C_l \langle \rho \rangle^{-5-l} \langle k\rho \rangle^{-2}, \\ |\partial_\rho^l \partial_k^2 V_{11}(\rho, k)| &\leq C_l \langle \rho \rangle^{-4-l} \langle k\rho \rangle^{-3} \ln \left(\frac{1}{k\rho} + 2 \right), \\ |\partial_\rho^l \partial_k^m V_{j2}(\rho, k)| &\leq C_l \langle \rho \rangle^{-3-l+m} \langle k\rho \rangle^{-1-m}, \quad j = 1, 2, \quad m = 0, 1, \\ |\partial_\rho^l \partial_k^2 V_{22}(\rho, k)| &\leq C_l \langle \rho \rangle^{-1-l} \langle k\rho \rangle^{-3} \ln \left(\frac{1}{k\rho} + 2 \right), \end{aligned}$$

Writing for z the following integral equation

$$z(\rho, k) = e^{ik\rho} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_{\rho}^{\infty} \begin{pmatrix} \frac{\sin k(\rho-s)}{k} & 0 \\ 0 & e^{-k(s-\rho)} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} z(s, k) ds,$$

and taking into account (4.6), one proves easily the existence of a smooth solution satisfying

$$(4.7) \quad |\partial_{\rho}^l \partial_k^m (e^{-ik\rho} z_1 - 1)| + \langle \rho \rangle |\partial_{\rho}^l \partial_k^m (e^{-ik\rho} z_2)| \leq C_l \langle \rho \rangle^{-2-l+m} \langle k\rho \rangle^{-1-m}, \quad m = 0, 1,$$

$$|\partial_{\rho}^n \partial_k^2 (e^{-ik\rho} z_1 - 1)| + |\partial_{\rho}^n \partial_k^2 (e^{-ik\rho} z_2)| \leq C \ln \left(\frac{1}{k\rho} + 2 \right), \quad n = 0, 1,$$

for all $\rho \geq R$, $k > 0$, $l \geq 0$.

To reconstruct f_1 , we set

$$z_0(\rho, k) = \int_R^{\rho} \frac{z_2(s, k)}{f_{3,2}(s, k)} ds - \int_R^{+\infty} \frac{z_2(s, 0)}{f_{3,2}(s, 0)} ds.$$

Then, for $\rho \geq R$, the statement of Lemma 4.2 follows directly from (4.7) and Lemma 4.1. To cover the case $x \leq R$ one can invoke the Cauchy problem with initial data at $\rho = R$. \square

Note that since $k^2 \in \mathbb{R}$, $f_2(\cdot, k) = \overline{f_1(\cdot, k)}$ is also a solution of (4.2).

Remark 4.3. Recall that the equation $\tilde{H}f = 0$ has a basis of explicit solutions $\rho\Phi_{\pm}(\rho)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$, $\rho\Theta_{\pm}(\rho)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$, with Φ_{\pm} , Θ_{\pm} given by (2.14). Comparing the behavior of $\rho\Phi_{\pm}$, $\rho\Theta_{\pm}$, with the asymptotics of $f_1(\rho, 0)$, $f_3(\rho, 0)$, one gets

$$(4.8) \quad f_1(\rho, 0) = \frac{1}{2}\rho(\xi_0(\rho) + \xi_1(\rho)), \quad f_3(\rho, 0) = \frac{1}{2}\rho(\xi_1(\rho) - \xi_0(\rho)),$$

where $\xi_0 = \frac{1}{\sqrt{3}}W\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\xi_1 = -\frac{2}{\sqrt{3}}W\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Next, we construct an exponentially growing solution at $+\infty$.

Lemma 4.4. *For any $k > 0$, there exists a solution $f_4(\rho, k)$ to (4.2) such that $f_4 = e^{k\rho}\chi_4$ with χ_4 verifying*

$$\partial_{\rho}^l (\chi_4(\rho, k) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = O_k(\rho^{-3-l}), \quad \rho \rightarrow +\infty.$$

Proof. We construct f_4 by means of the following integral equation:

$$(4.9) \quad \chi_4(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{\rho}^{+\infty} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} V\chi_4(s, k) ds$$

$$+ \int_{R_1}^{\rho} \begin{pmatrix} \frac{e^{k(s-\rho)} \sin k(\rho-s)}{k} & 0 \\ 0 & \frac{e^{2k(s-\rho)}}{2k} \end{pmatrix} V\chi_4(s, k) ds.$$

For $k > 0$ and R_1 sufficiently large (depending on k), the operator generating (4.9) is small on the space of bounded continuous functions. Therefore, (4.9) has a solution χ_4 verifying $|\chi_4(\rho, k)| \leq C$, $\rho \geq R_1$. Iterating this bound one gets that $\chi_4(\rho, k) - \binom{0}{1} = O_k(\rho^{-3})$ as $\rho \rightarrow \infty$. Finally, the estimates for the derivatives can be obtained by differentiating (4.9). \square

We now briefly describe some properties of the solutions f_j , $j = 1, \dots, 4$, that we will need later. Recall that the Wronskian $w(f, g) = \langle f', g \rangle_{\mathbb{R}^2} - \langle f, g' \rangle_{\mathbb{R}^2}$ does not depend on ρ if f and g are solutions of (4.1).

The estimates of Lemmas 4.1, 4.2, 4.4 lead to the relations:

$$(4.10) \quad w(f_1, f_2) = 2ik, \quad w(f_1, f_3) = w(f_2, f_3) = 0, \quad w(f_3, f_4) = -2k, \quad k > 0,$$

the three first relations being valid for $k = 0$ as well. Notice also that by Lemmas 4.1, 4.2, $\partial_k f_1(\rho, 0)$, $\partial_k f_3(\rho, 0)$ are solutions of the equation $\tilde{H}f = 0$ verifying for $\rho \geq 0$,

$$\begin{aligned} |\partial_k f_1(\rho, 0) - \binom{i\rho}{0}| &\leq C, \quad |\partial_{k\rho}^2 f_1(\rho, 0) - \binom{i}{0}| \leq \frac{C}{<\rho>^2}, \\ |\partial_k f_3(\rho, 0) + \binom{0}{\rho}| &\leq \frac{C}{<\rho>}, \quad |\partial_{k\rho}^2 f_3(\rho, 0) + \binom{0}{1}| \leq \frac{C}{<\rho>^2}, \end{aligned}$$

As a consequence, one has

$$(4.11) \quad \begin{aligned} w(\partial_k f_1|_{k=0}, f_1|_{k=0}) &= i, \quad w(\partial_k f_1|_{k=0}, f_3|_{k=0}) = 0, \\ w(\partial_k f_3|_{k=0}, f_1|_{k=0}) &= 0, \quad w(\partial_k f_3|_{k=0}, f_3|_{k=0}) = -1. \end{aligned}$$

In addition to scalar Wronskian we will use matrix Wronskians. If F, G are 2×2 matrix solutions of (4.2), their matrix Wronskian

$$W(F, G) = F^{t'}G - F^tG'$$

is independent of ρ .

Set $g_j(\rho, k) = f_j(-\rho, k)$, $j = 1, \dots, 4$. Since the potential V is even, g_j , $j = 1, \dots, 4$, are again solutions of (4.2) which have the same asymptotic behavior as $\rho \rightarrow -\infty$ as f_j as $\rho \rightarrow +\infty$.

Consider the matrix solutions F, G , defined by

$$F = (f_1, f_3), \quad G = (g_1, g_3).$$

Denote $D(k) = W(F, G)$. It follows from Lemmas 4.1, 4.2 that D is smooth for $k > 0$ and admits the estimate

$$(4.12) \quad |\partial_k^2 D(k)| \leq C \ln \left(\frac{1}{k} + 1 \right), \quad 0 < k \lesssim 1.$$

In addition, by (4.8), (4.10), (4.11), one has

$$(4.13) \quad D(0) = 0, \quad \partial_k D(0) = \begin{pmatrix} -2i & 0 \\ 0 & 2 \end{pmatrix}.$$

4.2. Scattering solutions and the distorted Fourier transform in a vicinity of zero energy. Set

$$(4.14) \quad \mathcal{F}(\rho, k) = F(\rho, k)s(k),$$

where $s(k) = D^{t-1}(k) \begin{pmatrix} 2ik \\ 0 \end{pmatrix}$. By (4.12), (4.13), $s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ is a smooth function of k for $0 < k < k_0$ (k_0 sufficiently small), continuous up to $k = 0$, verifying

$$(4.15) \quad \begin{aligned} s_1(0) &= -1, \quad s_2(0) = 0, \\ |\partial_k s(k)| &\leq C |\ln k|, \quad 0 < k \leq k_0. \end{aligned}$$

By construction, one has

$$w(\mathcal{F}, g_1) = 2ik, \quad w(\mathcal{F}, g_3) = 0,$$

for any $0 \leq k < k_0$. As a consequence,

$$(4.16) \quad \mathcal{F}(\rho, k) = r_1(k)g_1(\rho, k) + g_2(\rho, k) + r_2(k)g_3(\rho, k), \quad 0 \leq k < k_0,$$

with some coefficients $r_1(k)$, $r_2(k)$ that, by (4.8), (4.15), verify

$$(4.17) \quad r_1(0) = r_2(0) = 0.$$

Computing the Wronskians $w(\mathcal{F}, \bar{\mathcal{F}})$ and $w(\mathcal{F}, \bar{\mathcal{G}})$, where $\mathcal{G}(\rho, k) = \mathcal{F}(-\rho, k)$, one gets

$$|s_1(k)|^2 + |r_1(k)|^2 = 1, \quad r_1(k)\overline{s_1(k)} + \overline{r_1(k)}s_1(k) = 0, \quad 0 \leq k < k_0.$$

One can write the following Wronskian representation for r_1 :

$$(4.18) \quad r_1(k) = s_1(k) \frac{w(g_2, f_1)}{2ik} + s_2(k) \frac{w(g_2, f_3)}{2ik}, \quad k \neq 0.$$

Using (4.15) and the relations

$$w(g_2, f_3)|_{k=0} = w(g_2, f_1)|_{k=0} = \partial_k w(g_2, f_1)|_{k=0},$$

one easily deduces from (4.18) that r_1 is smooth for $0 < k < k_0$, continuous up to $k = 0$, and verifies

$$(4.19) \quad |\partial_k r_1(k)| \leq C |\ln k|, \quad 0 < k < k_0,$$

which in its turn, implies that r_2 is smooth for $0 < k < k_0$, continuous up to $k = 0$ and admits a similar estimate:

$$(4.20) \quad |\partial_k r_2(k)| \leq C |\ln k|, \quad 0 < k < k_0.$$

Introduce the following odd solution of (4.2):

$$e(\rho, k) = \mathcal{F}(-\rho, k) - \mathcal{F}(\rho, k).$$

By (4.14), (4.16),

$$(4.21) \quad e = a_1 f_1 + f_2 + a_2 f_3, \quad a_j = r_j - s_j, \quad j = 1, 2.$$

It follows from (4.15), (4.17), (4.19), (4.20) that

$$(4.22) \quad a_1(0) = 1, \quad a_2(0) = 0,$$

and

$$(4.23) \quad |\partial_k a_j| \leq C |\ln k|, \quad 0 < k < k_0, \quad j = 1, 2,$$

which together with Lemmas 4.1, 4.2 implies the following result.

Lemma 4.5. *One has:*

(i) $e(\rho, k) = e_0(\rho, k) + e_1(\rho, k)$, where $e_0(\rho, k) = a_1(k)e^{ik\rho} \binom{1}{0} + e^{-ik\rho} \binom{1}{0}$ and the remainder $e_1(\rho, k)$ admits the estimates

$$(4.24) \quad \begin{aligned} |e_1(\rho, k)| &\leq C(\langle \rho \rangle^{-2} + k |\ln k| e^{-k\rho}), \quad \rho \geq 0, \\ |\partial_k e_1(\rho, k)| &\leq C |\ln k| (\langle \rho \rangle^{-1} + e^{-k\rho/2}), \quad \rho \geq 0, \\ \|e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} &\leq C, \\ \|\rho e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} + \|\partial_k e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} &\leq C k^{-1/2} |\ln k|, \end{aligned}$$

for any $0 < k \leq k_0$.

(ii) $(\rho \partial_\rho - k \partial_k) e(\rho, k) = e^{ik\rho} \binom{1}{0} k \partial_k a_1(k) + e_2(\rho, k)$, with $e_2(\rho, k)$ verifying

$$(4.25) \quad \begin{aligned} |e_2(\rho, k)| &\leq C(\langle \rho \rangle^{-1} + k |\ln k| e^{-k\rho/2}), \quad \rho \geq 0, \\ \|e_2(\cdot, k)\|_{L^2(\mathbb{R}_+)} &\leq C, \end{aligned}$$

for any $0 < k \leq k_0$.

For $0 < \kappa \leq k_0$, introduce the operators $\mathbb{E}_\kappa : L^2(\mathbb{R}_+, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^2)$,

$$(\mathbb{E}_\kappa \Phi)(y) = \frac{1}{2^{3/2}\pi} \int_{\mathbb{R}_+} dk \theta_\kappa(k) \mathcal{E}(y, k) \Phi(k), \quad \Phi \in L^2(\mathbb{R}_+, \mathbb{C}^2),$$

where $\mathcal{E}(y, k)$ is a 2×2 matrix given by

$$\mathcal{E}(y, k) = \rho^{-1}(e(\rho, k), \sigma_1 \overline{e(\rho, k)}), \quad \rho = |y|,$$

$$\theta_\kappa(k) = \theta(\kappa^{-1}k), \quad \theta \text{ is a } C^\infty \text{ even function verifying } \theta(k) = \begin{cases} 1 & \text{if } |k| \leq 1/4 \\ 0 & \text{if } |k| \geq 1/2 \end{cases}.$$

Since $e(\rho, k)$ is a solution of the equation $\tilde{H}e = k^2 e$, one has $H\mathbb{E}_\kappa = \mathbb{E}_\kappa k^2 \sigma_3$.

By Lemma 4.5 (i), the operators \mathbb{E}_κ are bounded uniformly with respect to $\kappa \leq k_0$. The action of the adjoint operators $\mathbb{E}_\kappa^* : L^2(\mathbb{R}^3, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}_+, \mathbb{C}^2)$ is given by

$$(\mathbb{E}_\kappa^* \psi)(k) = \frac{1}{2^{3/2}\pi} \theta_\kappa(k) \int_{\mathbb{R}^3} dy \mathcal{E}^*(y, k) \psi(y), \quad \psi \in L^2(\mathbb{R}^3, \mathbb{C}^2).$$

Clearly,

$$(4.26) \quad \mathbb{E}_\kappa^* \sigma_3 \zeta_\pm = 0$$

for any $0 < \kappa \leq k_0$.

The following relation is a standard consequence of the asymptotics given by Lemma 4.5 (i),

$$(4.27) \quad \mathbb{E}_{\kappa_2}^* \sigma_3 \mathbb{E}_{\kappa_1} \sigma_3 = \theta_{\kappa_1}(k) \theta_{\kappa_2}(k),$$

for any $0 < \kappa_1, \kappa_2 \leq k_0$.

Remark 4.6. Notice that because of the presence of the cut off function θ_κ , \mathbb{E}_κ is bounded as an operator from $L^2([0, k_0])$ to $H^m(\mathbb{R}^3)$ for any $m \geq 0$, uniformly in $\kappa \leq k_0$.

We next introduce quasi-resonant functions $h_\kappa(y)$, $0 < \kappa \leq k_0$, by setting

$$h_\kappa = \sqrt{2} \mathbb{E}_\kappa \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Lemma 4.7. *For any $0 < \kappa \leq k_0$, $h_\kappa \in \langle y \rangle^{-1} L^2(\mathbb{R}^3)$ and as $\kappa \rightarrow 0$, one has*

$$(4.28) \quad \|h_\kappa\|_{L^2(\mathbb{R}^3)} = O(\kappa^{1/2}), \quad \|y h_\kappa\|_{L^2(\mathbb{R}^3)} = O(\kappa^{-1/2}),$$

$$(4.29) \quad \langle h_\kappa, \sigma_3(\xi_0 + \xi_1) \rangle = 4\pi + O(\kappa^{1/2} \ln \kappa), \quad \langle h_\kappa, \sigma_3(\xi_1 - \xi_0) \rangle = O(\kappa^{1/2} \ln \kappa).$$

Proof. Applying Lemma 4.5 (i), we decompose h_κ as follows:

$$(4.30) \quad \begin{aligned} h_\kappa(y) &= h_{\kappa,0}(y) + h_{\kappa,1}(y) + h_{\kappa,2}(y), \\ h_{\kappa,0}(y) &= \frac{1}{2\pi\rho} \kappa \hat{\theta}(\kappa\rho) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ h_{\kappa,1}(y) &= \frac{1}{2\pi\rho} \int_{\mathbb{R}_+} dk e^{ik\rho} (a_1(k) - 1) \theta_\kappa(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ h_{\kappa,2}(y) &= \frac{1}{2\pi\rho} \int_{\mathbb{R}_+} dk \theta_\kappa(k) e_1(\rho, k), \end{aligned}$$

where $\hat{\theta}(\rho) = \int_{\mathbb{R}} e^{ik\rho} \theta(k) dk$, $\rho = |y|$.

Clearly, $h_{\kappa,0} \in \langle y \rangle^{-1} L^2(\mathbb{R}^3)$ and one has

$$(4.31) \quad \|h_{\kappa,0}\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{1/2}, \quad \|y h_{\kappa,0}\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2}.$$

Consider $h_{\kappa,i}$, $i = 1, 2$. It follows from (4.22), (4.23), (4.24) that

$$(4.32) \quad \|h_{\kappa,i}\|_{L^2(\mathbb{R}^3)} \leq C\kappa, \quad \|y h_{\kappa,i}\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{1/2} |\ln \kappa|, \quad i = 1, 2,$$

which together with (4.31) leads to the estimates

$$(4.33) \quad \|h_\kappa\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{1/2}, \quad \|y h_\kappa\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2}.$$

We next compute $\langle h_\kappa, \sigma_3(\xi_1 \pm \xi_0) \rangle$. By (4.31), (4.32), as $\kappa \rightarrow 0$, one has

$$(4.34) \quad \begin{aligned} \langle h_\kappa, \sigma_3(\xi_1 \pm \xi_0) \rangle &= \langle h_{\kappa,0}, \sigma_3(\xi_1 \pm \xi_0) \rangle + O(\kappa^{1/2} \ln \kappa), \\ \langle h_{\kappa,0}, \sigma_3(\xi_1 - \xi_0) \rangle &= O(\kappa), \\ \langle h_{\kappa,0}, \sigma_3(\xi_1 + \xi_0) \rangle &= 2\kappa \int_{\mathbb{R}} d\rho \hat{\theta}(\kappa\rho) + O(\kappa) = 4\pi + O(\kappa), \end{aligned}$$

which gives (4.29). \square

4.3. Proof of Proposition 3.1. We start by deriving some coercivity bounds for the operator H .

Lemma 4.8. *There exists κ_0 , $0 < \kappa_0 \leq k_0$, and $C > 0$ such that*

$$(4.35) \quad \langle Hf, \sigma_3 f \rangle \geq C\kappa \|\nabla f\|_{L^2(\mathbb{R}^3)}^2,$$

for any $0 < \kappa \leq \kappa_0$ and any $f \in \dot{H}_{rad}^1(\mathbb{R}^3, \mathbb{C}^2)$ verifying

$$(4.36) \quad \langle f, \sigma_3 \zeta_- \rangle = \langle f, \sigma_3 \zeta_+ \rangle = \langle f, \sigma_3 h_\kappa \rangle = \langle f, \sigma_3 \sigma_1 \bar{h}_\kappa \rangle = 0.$$

Remark 4.9. Notice that since $\zeta_\pm, h_\kappa \in \langle y \rangle^{-1} L^2(\mathbb{R}^3)$ the scalar products that appear in (4.36) are well defined for any $f \in \dot{H}^1$.

Proof. The proof of Lemma 4.8 is based on the following result which is due to Duyckaerts and Merle:

Lemma 4.10. *There exists $c_0 > 0$ such that*

$$\langle Hf, \sigma_3 f \rangle \geq c_0 \|\nabla f\|_{L^2(\mathbb{R}^3)}^2,$$

for any $f \in \dot{H}_{rad}^1(\mathbb{R}^3, \mathbb{C}^2)$ verifying

$$\langle f, \sigma_3 \zeta_- \rangle = \langle f, \sigma_3 \zeta_+ \rangle = \langle f, \Delta \xi_0 \rangle = \langle f, \Delta \xi_1 \rangle = 0,$$

see [5] for the proof.

Let $f \in \dot{H}_{rad}^1$ such that (4.36) holds. One can write f as

$$f = \alpha_0 \xi_0 + \alpha_1 \xi_1 + g,$$

where

$$\alpha_j = -\frac{\langle f, \Delta \xi_j \rangle}{\|\nabla \xi_j\|_{L^2(\mathbb{R}^3)}^2}, \quad j = 0, 1,$$

and $g \in \dot{H}_{rad}^1$ verifies

$$\langle g, \sigma_3 \zeta_- \rangle = \langle g, \sigma_3 \zeta_+ \rangle = \langle g, \Delta \xi_0 \rangle = \langle g, \Delta \xi_1 \rangle = 0.$$

Therefore, by Lemma 4.10,

$$(4.37) \quad \langle Hf, \sigma_3 f \rangle = \langle Hg, \sigma_3 g \rangle \geq c_0 \|\nabla g\|_{L^2(\mathbb{R}^3)}^2.$$

Furthermore, since f verifies (4.36), one has

$$A(\kappa) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \langle g, \sigma_3 h_\kappa \rangle \\ \langle g, \sigma_3 \sigma_1 \bar{h}_\kappa \rangle \end{pmatrix},$$

where

$$A(\kappa) = - \begin{pmatrix} \langle \xi_0, \sigma_3 h_\kappa \rangle & \langle \xi_1, \sigma_3 h_\kappa \rangle \\ \langle h_\kappa, \sigma_3 \xi_0 \rangle & -\langle h_\kappa, \sigma_3 \xi_1 \rangle \end{pmatrix}.$$

By (4.29),

$$A(\kappa) = -2\pi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + O(\kappa^{1/2} \ln \kappa), \quad \kappa \rightarrow 0.$$

Therefore, for κ sufficiently small, one has

$$|\alpha_0| + |\alpha_1| \leq C \|\nabla g\|_{L^2(\mathbb{R}^3)} < y > h_\kappa\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2} \|\nabla g\|_{L^2(\mathbb{R}^3)}.$$

As a consequence,

$$\|\nabla f\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2} \|\nabla g\|_{L^2(\mathbb{R}^3)}.$$

Combining this inequality with (4.37) we get (4.35). \square

Next, we prove

Lemma 4.11. *There exists κ_1 , $0 < \kappa_1 \leq k_0$, and $C > 0$ such that for any $0 < \kappa \leq \kappa_1$ one has*

$$\|f\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{\kappa} \|\nabla f\|_{L^2(\mathbb{R}^3)},$$

for all $f \in H_{rad}^1(\mathbb{R}^3)$ verifying $\mathbb{E}_\kappa^* f = 0$.

Proof. By (4.22), (4.23) and Lemma 4.5 (i), $\mathbb{E}_\kappa^* f$ can be written as

$$(\mathbb{E}_\kappa^* f)(k) = \Phi_0(k) + \Phi_r(k),$$

where

$$\Phi_0(k) = \frac{1}{2^{3/2}\pi} \theta_\kappa(k) \check{f}(k),$$

$\check{f}(k) = 2 \int_{\mathbb{R}^3} dy \frac{\cos k|y|}{|y|} f(y)$, and the remainder Φ_r satisfies

$$\|\Phi_r\|_{L^2(\mathbb{R}_+)} \leq C\kappa^{1/2} \|f\|_{L^2(\mathbb{R}^3)}.$$

Therefore, $\mathbb{E}_\kappa^* f = 0$ implies

$$(4.38) \quad \|\check{f}\|_{L^2(0, \kappa/4)} \leq C\kappa^{1/2} \|f\|_{L^2(\mathbb{R}^3)}.$$

Notice also that for any $f \in H_{rad}^1$ and any $0 < \kappa \leq 1$ one has

$$\|f\|_{H^1(\mathbb{R}^3)} \leq C(\|\check{f}\|_{L^2(0, \kappa/4)} + \kappa^{-1} \|\nabla f\|_{L^2(\mathbb{R}^3)}).$$

Combining this inequality with (4.38), we get

$$\|f\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{\kappa} \|\nabla f\|_{L^2(\mathbb{R}^3)},$$

provided κ is sufficiently small. \square

We finally combine Lemmas 4.8, 4.11 to derive the following result which will be in the heart of the proof of Prop. 3.1

Lemma 4.12. *There exists κ_2 , $0 < \kappa_2 \leq k_0$, and $C > 0$ such that for any $0 < \kappa \leq \kappa_2$ one has*

$$(4.39) \quad \langle Hf, \sigma_3 f \rangle \geq C\kappa^3 \|f\|_{H^1}^2 - \frac{\kappa}{C} \|\mathbb{E}_\kappa^* \sigma_3 f\|_{L^2(\mathbb{R}_+)}^2,$$

for any $f \in H_{rad}^1(\mathbb{R}^3, \mathbb{C}^2)$ verifying $\langle f, \sigma_3 \zeta_\pm \rangle = 0$.

Proof. Write $f = f_1 + f_2$, where $f_1 = \mathbb{E}_\kappa \sigma_3 \mathbb{E}_\kappa^* \sigma_3 f$ and $f_2 = f - f_1$. One clearly has

$$(4.40) \quad \|f_1\|_{H^1(\mathbb{R}^3)} \leq C \|\mathbb{E}_\kappa^* \sigma_3 f\|_{L^2(\mathbb{R}_+)}, \quad \|Hf_1\|_{L^2(\mathbb{R}^3)} \leq C\kappa^2 \|\mathbb{E}_\kappa^* \sigma_3 f\|_{L^2(\mathbb{R}_+)},$$

for any $0 < \kappa \leq k_0$.

Consider f_2 . It follows from (4.26), (4.27) that for any $\kappa' \leq \kappa/2$,

- $\langle f_2, \sigma_3 \zeta_\pm \rangle = 0$;
- $\mathbb{E}_{\kappa'}^* \sigma_3 f_2 = 0$;
- $\langle f_2, \sigma_3 h_{\kappa'} \rangle = \langle f_2, \sigma_3 \sigma_1 \bar{h}_{\kappa'} \rangle = 0$.

Hence, by Lemmas 4.8, 4.11, one has

$$(4.41) \quad \langle Hf_2, \sigma_3 f_2 \rangle \geq C\kappa^3 \|f_2\|_{H^1(\mathbb{R}^3)}^2,$$

provided κ is sufficiently small.

Combining (4.40), (4.41) one gets (4.39). □

We are now in the position to prove Proposition 3.1. Consider the equation

$$(4.42) \quad \begin{aligned} i\psi_\tau &= P\mathcal{H}(\tau)P\psi, \\ \psi|_{\tau=s} &= f, \end{aligned}$$

where

$$\mathcal{H}(\tau) = H + \tau^{-1}l, \quad l = \alpha_1 \sigma_3 - i\nu_1 \left(\frac{1}{2} + y \cdot \nabla \right),$$

$\alpha_1, \nu_1 \in \mathbb{R}$, $s > 0$ and $f \in \mathcal{S}(\mathbb{R}^3)$ verifying $\langle f, \sigma_3 \zeta_\pm \rangle = 0$.

Fix κ such that $0 < \kappa \leq \kappa_2$ and consider the functional $G_1(\tau) = \langle H\psi, \sigma_3 \psi \rangle + c_0 \|\mathbb{E}_\kappa^* \sigma_3 \psi\|_{L^2(\mathbb{R}_+)}^2$. Clearly,

$$(4.43) \quad G_1(\tau) \leq C \|\psi(\tau)\|_{H^1(\mathbb{R}^3)}^2.$$

Moreover, since $\langle \psi(\tau), \sigma_3 \zeta_\pm \rangle = 0$, choosing c_0 sufficiently large, we get:

$$(4.44) \quad G_1(\tau) \geq c_1 \|\psi(\tau)\|_{H^1(\mathbb{R}^3)}^2.$$

We next compute the derivative $\frac{d}{d\tau} G_1$. One has

$$i \frac{d}{d\tau} \langle H\psi, \sigma_3 \psi \rangle = \frac{2i}{\tau} \operatorname{Im} \langle l\psi, \sigma_3 H\psi \rangle,$$

which implies

$$(4.45) \quad \left| \frac{d}{d\tau} \langle H\psi, \sigma_3 \psi \rangle \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\nabla \psi(\tau)\|_{L^2(\mathbb{R}^3)}^2.$$

Next, we address $\|\mathbb{E}_\kappa^* \sigma_3 \psi\|_{L^2(\mathbb{R}_+)}^2$. Denote $\Phi(\tau) = \mathbb{E}_\kappa^* \sigma_3 \psi(\tau)$. Then $\Phi(k, \tau)$ solves

$$(4.46) \quad i\Phi_\tau = k^2 \sigma_3 \Phi + \frac{1}{\tau} Y,$$

where

$$Y = \mathbb{E}_\kappa^* \sigma_3 l\psi.$$

Integrating by parts and applying Lemma 4.5 (ii), one can rewrite Y in the form

$$Y(k, \tau) = Y_0(k, \tau) + Y_1(k, \tau),$$

where

$$Y_0(k, \tau) = i\nu_1 k \partial_k \Phi(k, \tau),$$

and $Y_1(k, \tau)$ admits the estimate

$$\|Y_1(\tau)\|_{L^2(\mathbb{R}_+)} \leq C(|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{L^2(\mathbb{R}^3)}.$$

Therefore, (4.46) gives

$$\left| \frac{d}{d\tau} \|\Phi(\tau)\|_{L^2(\mathbb{R}_+)}^2 \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{L^2(\mathbb{R}^3)}^2.$$

Combining this inequality with (4.46) and taking into account (4.44) one gets

$$(4.47) \quad \left| \frac{d}{d\tau} G_1(\tau) \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{H^1(\mathbb{R}^3)}^2 \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) G_1(\tau).$$

Integrating we obtain

$$G_1(\tau) \leq C \left(\frac{s}{\tau} \right)^{C(|\alpha_1| + |\nu_1|)} G_1(s), \quad 0 < \tau \leq s,$$

which by (4.43), (4.44), leads to the bound

$$(4.48) \quad \|U(\tau, s)f\|_{H^1(\mathbb{R}^3)} \leq C \left(\frac{s}{\tau} \right)^{C(|\alpha_1| + |\nu_1|)} \|f\|_{H^1(\mathbb{R}^3)},$$

for any $0 < \tau \leq s$ and any $f \in H_{rad}^1$.

To control the higher regularity, consider the functional $G_2(\tau) = \langle H^2 \psi, \sigma_3 H \psi \rangle + c_2 G_1(\tau)$. One has

$$C^{-1} \|\psi\|_{H^3(\mathbb{R}^3)}^2 \leq G_2 \leq C \|\psi\|_{H^3(\mathbb{R}^3)}^2,$$

provided c_2 is chosen sufficiently large.

Computing the derivative $\frac{d}{d\tau} \langle H^2 \psi(\tau), \sigma_3 H \psi(\tau) \rangle$ and taking into account (4.47) we get

$$(4.49) \quad \left| \frac{d}{d\tau} G_2(\tau) \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{H^3(\mathbb{R}^3)}^2 \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) G_2(\tau).$$

which implies

$$(4.50) \quad \|U(\tau, s)f\|_{H^3(\mathbb{R}^3)} \leq C \left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)} \|f\|_{H^3(\mathbb{R}^3)},$$

for any $0 < \tau \leq s$.

The H^2 bounded stated in Prop. 3.1 follows from (4.48), (4.50) by interpolation.

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